REGULAR IMPULSIVE MOTIONS IN A ONE-DIMENSIONAL SYSTEM

(PRAVIL'NYE IMPUL'SNYE DVIZHENIIA V ODNOMERNOI SISTEME)

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Aim of this paper is the investigation of dynamics of a system comprising an arbitrary number of spheres interacting by means of direct and central but not completely elastic impacts. We assume that in the intervals between impacts each sphere has only one "linear" degree of freedom and moves in a force field obeying a known law. External perturbation which is also assumed known is nonindependent, and is applied to the extreme spheres of the system. The problem considered is of interest when investigating the behavior of a layer of granular substance in a vibrating vessel [1] and in connection with some other technical applications [2].

Statistical averaging of motion of the system and its consequent global treatment within the framework of statistical mechanics poses fundamental difficulties, since the system has no integral invariant and its phase volume is not constant. Therefore, in the following, a deterministic approach will be adopted and only small neighborhoods of periodic modes possessing well defined selected properties will be investigated. From the very beginning the problem will have to be severely restrained by specifying the character of motion under investigation very accurately. The class of regular motions studied below exhibits a characteristic feature; during the process of motion the i th sphere collides, in turn, with the preceding and the following sphere. Thus we exclude, for example, the situation when the ith sphere collides with the $(i-1)$-th sphere twice in succession. However, any other reasonable scheme of motion could be adopted.

Regular periodic motions are found by means of point transformations [3], but transformations themselves are treated as partial difference equations and form a starting point for subsequent, purely analytical treatment of the problem.

1. General formulation of the problem. We shall divide the process of regular motions taking place in the system, into intervals and each interval will characterize a single transmission of an impulse from the first, extreme left-hand sphere to the last, nth sphere. We shall denote the intervals by $k = 1, 2, \ldots, \infty$, but they will not be related to any period of time, and coincidence of intervals will not imply simultaneity in time. Regular motion on the kth sphere of the system ($i = 1, \ldots, n$) will be completely described by the following parameters: u_{ik} which is the initial velocity of a sphere after its collision with the $(i+1)$-th sphere during the kth transmission of impulse; v_{ik} which is the initial velocity of a sphere after its collision with the $(i-1)$th sphere during the kth transmission of impulse; τ_{ik} and t_{ik} which are the durations of collisionless motion of a sphere in the reverse and forward direction respectively.

Magnitudes of final velocities of the spheres i. e. of their velocities immediately before the collision and lengths of paths in forward and reverse directions can, by virtue of the integrability of equations of motion of a sphere in the intervals between collisions, be completely defined provided that the corresponding initial velocities and time intervals are given (1.1)

$$v_{ik+} = v_{ik+}(v_{ik},\ t_{ik}),\quad u_{ik+} = u_{ik+}(u_{ik},\ \tau_{ik}),\quad s_{ik} = s_{ik}(v_{ik},\ t_{ik}),\quad l_{ik} = l_{ik}(u_{ik},\ \tau_{ik})$$

Fig. 1 gives a schematic illustration of the process of regular motion.

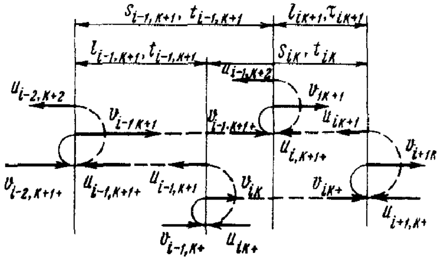

Fig. 1

Total time of transmission of a single impulse along the length of the system from left to right or, in other words, duration of an interval, is equal to

$$t_k = t_{1k} + \ldots + t_{nk}$$

All the quantities introduced previously must be basically nonnegative if regular motions are to take place in the system, otherwise the process of subdivision of motion into intervals becomes impossible. Moreover, other exact relations must also be fulfilled, namely the conditions of compatibility of motion in space and time (Fig. 1)

$$s_{ik} + l_{i-1,\,k+1} = l_{i,\,k+1} + s_{i-1,\,k+1},\qquad t_{ik} + \tau_{i,\,k+1} = \tau_{i-1,\,k+1} + t_{i-1,\,k+1} \tag{1.2}$$

Finite relations (1.1) and (1.2) together with equations describing an imperfectly elastic collision (1.3)

$$m_{i-1}v_{i-1,\,k+} - m_i u_{ik+} = -m_{i-1}u_{i-1,\,k+1} + m_i v_{ik},\quad R_i(v_{i-1,\,k+} + u_{ik+}) = v_{ik} + u_{i-1,\,k+1}$$

where m_i is the mass of the ith sphere and $0 < R_i < 1$ is the coefficient of restitution during the impact of the ith sphere on the $(i-1)$-th sphere, represent a closed system of nonlinear partial difference equations and define the motion completely, provided that the boundary conditions of motion of the extreme spheres of the system which are under direct influence of external forces, are known, together with the initial conditions characterizing the state of the system during the first transmission of impulse (i. e. during the first interval). In its general form however, this system can only be utilized for numerical methods. Analytical approach requires further simplification and more concise formulation of the problem.

2. Simplest periodic motions of a free system in a limited volume. If between the collisions spheres are in the state of inertial motion of constant velocity, then, in the following we shall call such a system of spheres, a free system. Regular motions in one-dimensional free system are, obviously, characterized by the following relations

$$u_{ik+} = u_{ik},\quad v_{ik+} = v_{ik},\qquad l_{ik} = u_{ik}\tau_{ik},\quad s_{ik} = v_{ik}t_{ik} \tag{2.1}$$

Let us assume that all spheres are identical, i. e.

$$m_i = m,\qquad R_i = R \tag{2.2}$$

and move along a straight line between two walls, whose mean separation is S. Right-hand side wall is fixed, while the left-hand wall oscillates harmonically as $a \sin \omega t$ where a and ω are, respectively, amplitude and frequency.

In this case (Fig. 2) the overall condition limiting the volume, will become (*)

*) Size of spheres can be disregarded without loss of generality.

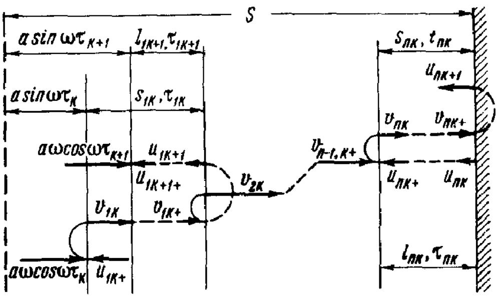

Fig. 2

$$S = a \sin \omega\tau_k + \sum_{j=1}^{n} s_{jk} \qquad (2.3)$$

where τ_k is the time of commencement of the k th interval.

Conditions of compatibility of motion of the first sphere in time and of the last sphere in space, will now be, respectively

$$\tau_{k+1} = \tau_k + t_{1k} + \tau_{1,k+1}, \qquad s_{nk} = l_{nk} \qquad (2.4)$$

and we find that the condition of compatibility of motion of the first sphere in space

$$a \sin \omega\tau_k + s_{1k} = a \sin \omega\tau_{k+1} + l_{1,\,k+1} \qquad (2.5)$$

is a derived one and is obtained by the summation of first equations of (1. 2) over i, with (2. 3) and (2. 5) taken into account.

Finally, equations of impact of extreme spheres on the walls can be written as

$$R\,(u_{1k+} + a\omega \cos \omega\tau_k) = v_{1k} - a\omega \cos \omega\tau_k, \quad R v_{nk+} = u_{n,k+1} \qquad (2.6)$$

and we can assume that the coefficient of restitution in the collision with a wall is the same as that in collision with another sphere.

In the simplest periodic solution of the discrete boundary value problem formulated above we find, that the physical characteristics of motion of the spheres do not change from one interval to another and that the times at which two neighboring intervals commence, differ from each other by an amount equal to the period of motion

$$T = 2\pi\nu \,/\, \omega \qquad (\nu = 1, 2\ldots)$$

where ν is the multiplicity of the mode. In other words,

$$v_{ik} = v_i, \quad u_{ik} = u_i, \quad t_{ik} = t_i, \quad \tau_{ik} = \tau_i, \quad \tau_k = (2\pi\nu k + \varphi)\,/\,\omega \qquad (2.7)$$

where φ is the constant phase of impact of the first sphere on the left-hand wall. Now we shall proceed to find the mode of motion, noting first that comparison of (1. 2) and (2. 4) gives, together with (2. 7)

$$t_i + \tau_i = 2\pi\nu \,/\, \omega, \quad s_i = l_i \quad (i = 1, \ldots, n) \qquad (2.8)$$

Thus each sphere performs periodic oscillations of period T and amplitude s_i. Further, the following expressions for the times of forward and reverse motions are obtained:

$$t_i = \frac{2\pi\nu}{\omega}\,\frac{u_i}{u_i + v_i}, \qquad \tau_i = 2\,\frac{2\pi\nu}{\omega}\,\frac{v_i}{u_i + v_i} \qquad (2.9)$$

Equations of conservation of momentum during the impact (first group of relations of (1. 3)) together with conditions (2. 1) ,(2. 2) and (2. 7) show that the sum of the moduli of forward and reverse velocities is constant and the same for every sphere

$$v_i + u_i = v \qquad (2.10)$$

Let us represent the second group of relations of (1. 3) as a linear, first order finite difference equation

$$w_i - w_{i-1} = -2\,\frac{1-R}{1+R}\,v \qquad (w_i = v_i - u_i) \qquad (2.11)$$

We easily see that solution of this equation satisfying the second condition of (2. 6) is of the form

$$w_i = [1 + 2(n - i)] \frac{1 - R}{1 + R} v \tag{2.12}$$

First equation of (2, 6) will now yield the value of v

$$v = \frac{1}{n} \frac{1 + R}{1 - R} a\omega \cos \varphi \tag{2.13}$$

Finally, comparing relations (2, 10) and (2, 13) we arrive at expressions for the forward and reverse velocities

$$v_i = \left[\frac{1}{n} \frac{1}{1 - R} + 1 - \frac{i}{n}\right] a\omega \cos \varphi$$

$$(i = 1, \ldots, n) \quad (2.14)$$

$$u_i = \left[\frac{1}{n} \frac{R}{1 - R} - 1 + \frac{i}{n}\right] a\omega \cos \varphi$$

Condition of the positiveness of forward velocities of spheres ($v_1 > 0$) limits the phase of impact of the first sphere on the wall to $-\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi$. Physically it means that, at the moment of impact, the wall should be moving in the positive (left to right) direction. Condition of the positiveness of reverse velocities ($u_{min} = u_1 > 0$) leads to the definition of a domain of existence of the simplest mode of motion, in terms of the coefficient of restitution

$$1 - 1/n < R < 1 \tag{2.15}$$

from which we see that the domain narrows with the increasing number of spheres.

Thus, velocities of forward motions diminish continuously, while velocities of reverse motions continuously increase from one sphere to another according to a linear law, preserving however at all times, the relation $v_{min} > u_{max}$ (Fig. 3).

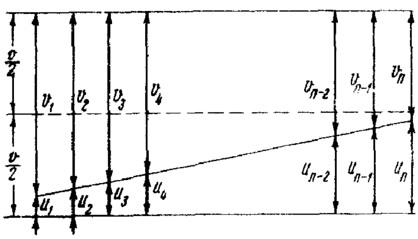

Fig. 3

We should note that the pattern of change of durations of forward and reverse motions of spheres along the system is, as seen from (2, 9), quite different. Free paths of the spheres

$$s_i = \frac{2\pi\nu}{\omega} \frac{u_i v_i}{v}$$

change in a parabolic manner and the minimum density of spheres (maximum free paths) is maintained near the fixed, right-hand wall.

Simplest modes of motion of the considered system have a characteristic property, consisting of the fact that the impulse

$$I_i = I = mv \tag{2.16}$$

transmitted by the spheres up to the fixed wall is constant. Expression

$$= \sum_{i=1}^{n} t_i = \frac{2\pi\nu}{\omega} n \frac{1 + R}{1 - R} \left[\frac{R}{1 - R} - \frac{n - 1}{2}\right]$$

gives the time of transmission of an impulse from one wall to the other.

Taking into account relations (2, 3) and (2, 9) we can further write expressions for the mean (over one period) force acting on the fixed wall (or on every ith sphere from the direction of the $(i-1)$-th sphere) and the kinetic energy of spheres. These are, respectively

$$F = \frac{\omega}{2\pi\nu} I = c(S - a \sin \varphi) \tag{2.17}$$

$$E = \frac{\omega}{2\pi\nu}\frac{m}{2}\sum_{j=1}^{n}(v_j^2 t_j + u_j^2 \tau_j) = \frac{1}{2}c(S - a\sin\varphi)^2 \qquad (2.18)$$

$$c = \frac{m}{T^2 n}\frac{(1+R)^2}{R - {}^1/_3(n^2-1)(1-R)^2}$$

where c is definitely positive within the domain (2, 15) and has the dimension of rigidity. Thus the considered system behaves, on the average, as a linear spring of rigidity c, compressed (stretched) by the amount equal to the distance occupied by the moving spheres.

To make the simplest periodic mode in the system fully definable, it remains to define the phase φ of impact of the first sphere on the wall. Equation defining φ can be obtained from the condition of finiteness of the volume (2, 3) which, after a simple summation yields (*) (2.19)

$$\frac{S}{a} = \sin\varphi + 2\pi\nu f_n(R)\cos\varphi \quad \left(f_n(R) = \frac{1}{1+R}\left[\frac{R}{1-R} - \frac{n^2-1}{3}(1-R)\right]\right)$$

where f_n is a positive magnitude increasing monotonously with R within (2, 15). Moreover, we obviously have

$$f_n(R) > f_n(1 - 1/n) = {}^1/_3(n-1)$$

Equation (2, 19) allows solutions only if

$$S/a < [1 + (2\pi\nu)^2 f_n(R)^2]^{1/2} \qquad (2.20)$$

Conditions (2, 20) and (2, 15) together with the obvious inequality

$$S/a > 1 \qquad (2.21)$$

fully define the domain of existence of simplest modes in the parametric space of the system. If we now introduce the notation

$$\sin\gamma = S/a\,[1 + (2\pi\nu)^2 f_n(R)^2]^{-1/2}, \qquad 0 < \gamma < {}^1/_2\pi \qquad (2.22)$$

$$\cos\delta = [1 + (2\pi\nu)^2 f_n(R)^2]^{-1/2}, \qquad 0 < \delta < \cos^{-1}[1 + (2\pi\nu)^2\, {}^1/_9(n-1)^2]^{1/2}$$

then two basically different solutions of (2, 19) possessing a physical meaning within the domain of existence can be represented by

$$\varphi_1 = \gamma - \delta, \quad \varphi_2 = \pi - (\gamma + \delta) \qquad (2.23)$$

Since by (2, 2) $\sin\gamma > \cos\delta$ and consequently $\gamma + \delta > \frac{1}{2}\pi$, phases φ_1 and φ_2 of these modes vary over the nonintersecting intervals

$$0 < \varphi_2 < {}^1/_2\pi, \qquad -{}^1/_2\pi < \varphi_1 < \varphi_2 \qquad (2.24)$$

We should note that the conditions of positiveness of the forward velocities of spheres is fulfilled automatically. Both modes approach each other if $\gamma \to \frac{1}{2}\pi$ and we reach the boundary of the domain of existence, given by (2, 20).

3. Stability of simplest periodic motions of the system. We shall base our investigations of the stability of the simplest mode of motion found in the previous Section on the fact, that the motion retains its regularity in the presence of weak perturbations and can therefore be still described by the same nonlinear partial difference system. Varying the independent discrete variables of the problem, we arrive at

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*) Relations (2, 17) and (2, 19) are given implicitly in [2], which also discusses quasielastic properties of the considered system.

a linear homogeneous system whose general solution can be given by superimposing particular solutions of the type

$$\delta v_{ik} = v x_i \mu^k, \qquad \delta u_{ik} = v y_i \mu^k, \qquad \delta t_{ik} = (2\pi\nu / \omega)\psi_i \mu^k$$
$$\delta\tau_{ik} = (2\pi\nu / \omega)\psi_i^* \mu^k, \qquad \delta\tau_k = (2\pi\nu / \omega)\sigma\mu^k \tag{3.1}$$

Total number of particular, mutually independent solutions of (3.1) and the corresponding number of the eigen numbers μ is equal to $2n$, i.e. to the order of initial system. Set of the eigen numbers describes the rate of rise or decay of perturbations in the system. The necessary and sufficient condition for the motion to be stable under weak perturbations is, that for any particular solution (3.1)

$$|\mu| < 1 \tag{3.2}$$

holds.

Dimensionless eigenforms $x_1, y_1, \psi_1, \psi_1^*$ and σ of mutually independent particular solutions (3.1), satisfy the following homogeneous linear difference system with variable discrete coefficients

$$v_i\psi_i + u_i x_i + \mu u_{i-1}\psi_{i-1}^* + \mu v_{i-1} y_{i-1} = \mu u_i \psi_i^* + \mu v_i y_i + \mu v_{i-1}\psi_{i-1} + \mu u_{i-1} x_{i-1}$$
$$\psi_i + \mu\psi_{i-1}^* = \mu(\psi_{i-1} + \psi_{i-1}^*) \tag{3.3}$$

$$x_i + y_i = x_{i-1} + \mu y_{i-1}, \qquad x_i + \mu y_{i-1} = R(x_{i-1} + y_i) \tag{3.4}$$

and with the following boundary conditions:

$$\psi_1 + \mu\psi_1^* = \sigma(\mu - 1), \qquad v_n\psi_n + u_n x_n = u_n\psi_n^* + v_n y_n \tag{3.5}$$

$$x_1 - R y_1 = -n(1 - R)2\pi\nu\sigma \tan\varphi, \quad \mu y_n - R x_n = 0 \tag{3.6}$$

Subsystem (3.3) is obtained by varying the variables under the condition of compatibility of the motion in space and time (1.2), while (3.4) is obtained by varying the variables in the equations of impact (1.3). Boundary conditions (3.5) and (3.6) are stipulated by (2.4) and (2.6), respectively.

To this we must add another unlocalized condition obtained by varying Equations (2.3) limiting the volume

$$a\omega\sigma\cos\varphi + \sum_{j=1}^{n}(v_j\psi_j + u_j x_j) = 0 \tag{3.7}$$

or an equivalent expression

$$(\mu - 1)a\omega\sigma\cos\varphi = v_1\psi_1 + u_1 x_1 - \mu(u_1\psi_1^* + v_1 y_1) \tag{3.8}$$

derived from (2.5).

Passing now to investigation of stability of the simplest modes of motion, we shall first define the aperiodic boundary of the region of stability, corresponding to $\mu = 1$. In this particular case we have, from (3.5),

$$\psi_1 + \psi_1^* = 0$$

while (3.3) yields

$$v_i\psi_i + u_i x_i = u_i\psi_i^* + y_i v_i, \qquad \psi_i + \psi_i^* = 0$$

Eigenforms of variations of velocities are found to be proportional to corresponding velocities

$$x_i / v_i = y_i / u_i = \text{const}$$

and since $v\psi_i = y_i v_i - x_i u_i$, the eigenforms of variations of time intervals become identically zero

$$\psi_i \equiv \psi_i^* \equiv 0 \tag{3.9}$$

This implies that the system exhibits only a velocity drift. For $\mu = 1$, condition (3.7) divided by $\sigma \neq 0$ will become

$$\cos\varphi - 2\pi\nu f_n(R)\sin\varphi = 0, \qquad \text{or} \qquad \frac{\partial}{\partial\varphi}\frac{S}{a} = 0 \tag{3.10}$$

Thus the aperiodic boundary of stability $\mu = 1$ coincides with the boundary of the domain of existence $\gamma = \frac{1}{2}\pi$ on which the periodic modes (2.23) merge. If $\mu \neq 1$, then the following variable substitution becomes possible

$$y_i = z_{i+1} - z_i, \qquad x_i = \mu z_1 - z_{i+1} \tag{3.11}$$

First group of Equations (3.4) is now satisfied identically, while the second group, after some transformations, can be written as a linear, homogeneous, second order difference system

$$z_{i+1} - 2\frac{\mu + R}{1 + R} z_i + \mu z_{i-1} = 0 \tag{3.12}$$

Similarly, after the following substitution of variables

$$\psi_i^* = \vartheta_i - \frac{\vartheta_{i+1} + x_i}{\mu}, \qquad \psi_i = \vartheta_{i+1} - \vartheta_i + y_i \tag{3.13}$$

first group of equations (3.3) is satisfied identically, while the second group, after some transformations and division by $v \neq 0$, becomes a linear inhomogeneous system

$$\vartheta_{i+1} - 2\frac{\mu + R}{1 - R}\vartheta_i + \mu\vartheta_{i-1} = \frac{\mu - 1}{1 + R}\Big[(1 - R)(n - i + 1)(z_{i+1} - \mu z_{i-1}) + \\ + (\mu - 1)\frac{2R}{1 + R} z_i\Big] \tag{3.14}$$

Boundary conditions (3.5) and (3.6) and mutually equivalent relations (3.7) and (3.8) now become

$$\vartheta_1 = \sigma, \qquad -\frac{\mu + R}{1 + R}\vartheta_{n+1} + \mu\vartheta_n = (\mu - 1)^2\frac{R}{(1 + R)^2} z_{n+1} \tag{3.15}$$

$$z_2 - \frac{\mu + R}{1 + R} z_1 = \sigma\frac{1 - R}{1 + R} n 2\pi\nu \tan\varphi, \qquad -\frac{\mu + R}{1 + R} z_{n+1} + \mu z_n = 0 \tag{3.16}$$

$$\sigma + \sum_{i=1}^{n}\Big[\Big(\frac{1}{n}\frac{1}{1 - R} + 1 - \frac{i}{n}\Big)(\vartheta_{i+1} - \vartheta_i) + \\ + \Big(\frac{1}{n}\frac{R}{1 + R} - 1 + \frac{i}{n}\Big)(\mu z_i - z_{i+1})\Big] = 0 \tag{3.17}$$

or

$$\sigma(\mu - 1) = \frac{1}{n}\frac{1 + R}{1 - R}(\vartheta_2 + \mu z_1) + \\ + \Big(\mu - 1 - \frac{1}{n}\frac{\mu + R}{1 - R}\Big)\vartheta_1 - \Big(\mu - 1 + \frac{1}{n}\frac{\mu + R}{1 - R}\Big) z_2 \tag{3.18}$$

We see now that, since the coefficients of the homogeneous part of the system (3.14) are constant, the process of investigation of the simplest modes can be brought to completion, for any n, by purely analytical means. Solution of (3.12) which, for $\mu \neq R^2$ (and naturally, for $\mu \neq 1$), is

$$z_i = C_1 h_1^{\,i} + C_2 h_2^{\,i} \qquad \Big(h_{1,2} = \frac{\mu + R \pm \sqrt{(\mu - 1)(\mu - R^2)}}{1 + R}\Big) \tag{3.19}$$

is found first. Here $h_{1,2}$ are the eigen numbers of the system, while C_1 and C_2 are constants definable in terms of σ given in the boundary conditions (3.16). Expression for the discrete variable ϑ_i will, by virtue of coincidence of homogeneous parts of the system (3.12) and (3.14), contain "secular" terms $ih_k^{\,i}$ and $i^2h_k^{\,i}$ ($k = 1, 2$) and will also depend on two constants definable in terms of σ from conditions (3.15). Eigen numbers are given by equation obtained from the conditions (3.17) and (3.18), and when divided by $\sigma \neq 0$, can be written as

$$P_{2n}(\mu) = 0$$

where $P_{2n}(\mu)$ is a certain $2n$th degree polynomial. For $n = 1$ say, this equation becomes

$$\mu^2 - [2R + (1 - R^2)2\pi\nu \operatorname{tg} \varphi]R\mu + R^4 = 0 \tag{3.20}$$

from which we see that the oscillatory periodic boundary of stability $N_-(\mu = -1)$ has, in case of one sphere, the form

$$\cos \varphi + \frac{R(1 - R^2)}{(1 + R^2)^2} 2\pi\nu \sin \varphi = 0$$

and can be attained within the domain of existence only in case of the mode, for which $\varphi = \varphi_1 < 0$. Quasiperiodic oscillating boundary $N_\psi(\mu = \exp j\psi)$ coincides with the boundary $R = 1$ of the domain of existence (here and in the following $j = \sqrt{-1}$).

Region of stability of the mode for which $\varphi = \varphi_1$ when $n = 1$ is given, by

$$\tan \varphi_1 > -\frac{1}{2\pi\nu} \frac{(1 + R^2)^2}{R(1 - R^2)}$$

and it gradually narrows with increasing ν.

Now we shall turn our attention to a particular case, when one of the eigen numbers of the mode of motion is $\mu = R^2$. Eigen numbers corresponding to $\mu = R^2$ are $h_1 = h_2 = R$ and the general solution of (3.12) contains a "secular" term

$$z_i = (C_1 + C_2 i)R^i \tag{3.21}$$

Boundary conditions (3.16) are satisfied only if $C_2 = 0$ and

$$\varphi = \varphi_1 = 0 \tag{3.22}$$

Thus we have here the case of stability of a symmetric mode of motion, when the first sphere hits the wall at an instant of its mean position. The following relation between the parameters of the system

$$S / a = 2\pi\nu f_n(R) \tag{3.23}$$

is necessary for the symmetric mode to exist.

To obtain a clear idea of the stability of symmetric mode, we must compute the remaining $2n-1$ eigen numbers (first eigen number of the mode is $\mu = R^2 < 1$). Let us therefore put $\varphi = \varphi_1 = 0$ and $\mu \neq R^2$. Then the system of equations (3.12) with boundary conditions (3.16) separates out of the general system and, consequently, the drift of velocity parameters can take place independently. Constants C_1 and C_2 in (3.19) satisfy the linear homogeneous system (3.16) and are, therefore, not simultaneously equal to zero, only if its determinant becomes zero, i. e. if

$$h_1^n = h_2^n, \quad \text{for} \quad \frac{\mu + R + [(\mu - 1)(\mu - R^2)]^{1/2}}{\mu + R - [(\mu - 1)(\mu - R^2)]^{1/2}} = \exp \frac{2\pi k j}{n} \tag{3.24}$$

which can also be given as

$$\mu - (1 + R) \cos \pi (k / n)\sqrt{\mu} + R = 0 \tag{3.25}$$

If

$$\left| \cos \pi \frac{k}{n} \right| < \frac{2\sqrt{R}}{1 + R} \tag{3.26}$$

then the eigen numbers given by (3.25) are complex and their moduli are

$$|\mu| = R < 1 \tag{3.27}$$

If on the other hand the inequality (3.27) is not filfilled, then the roots of (3.25) are real

$$\sqrt{\mu} = \frac{1+R}{2}\cos\pi\frac{k}{n} \pm \left(\frac{(1+R)^2}{4}\cos^2\pi\frac{k}{n} - R\right)^{1/2}$$

and satisfy the estimate

$$|\sqrt{\mu}| < {}^1/_2\,(1+R) + ({}^1/_4\,(1+R)^2 - R)^{1/2} = 1 \tag{3.28}$$

From this we see that perturbations of velocity characteristics of the symmetric mode does not lead to loss of stability. Instability can however be connected with the drift of temporal characteristics of the mode. To throw some light on the character of the temporal drift near the symmetric mode, let us put $z_1 = 0$. Then the solution of homogeneous equation (3.14) with boundary conditions (3.15) can be written as

$$\vartheta_i = \sigma\frac{h_1^{-n+i-1} + h_2^{-n+i-1}}{h_1^{-n} + h_2^{-n}} \tag{3.29}$$

Further, substituting Expression (3.29) with $z_1 = 0$ into (3.18) and after some transformations and division by $\sigma \neq 0$ we obtain

$$(h_1^{-n} - h_2^{-n})\ (h_1 - h_2) = 0$$

which shows at once that the eigen numbers of the temporal drift do not change.

Consequently, the symmetric mode of arbitrary multiplicity ν is always asymptotically stable for any number n of spheres, provided of course that conditions of existence in terms of the coefficient of restitution given by (2.15) are fulfilled.

The problem considered by us which dealt with simplest periodic motions of a free system in a restricted volume was, to a large extent, used to illustrate the method. Relations however, formulated in the general statement of the problem of regular motions, allow us to investigate in a localized setting, the dynamics of a whole class of one-dimensional systems of mutually colliding spheres.

Simplest motions of a free system under different boundary conditions can be sought in an analogous manner. For example, if both walls are rigidly connected and oscillate harmonically, then only the conditions (2.4) and (2.6) imposed on the motion of the nth sphere will change, together with the restriction of volume, which in this case will become

$$S = s_{1k} + \dots + s_{nk}$$

Simplest periodic mode characterized by the fact that

$$v_i = u_{n-i+1}$$

can be found without difficulty.

We can also relax the restriction of volume and assume e.g. that the last sphere of our system (of an arbitrary mass) is acted upon by a constant force in the negative direction. Generalization to the case of a wall oscillating according to any harmonic law is, in general, of trivial character.

Other finite difference equations are obtained when more complex regular periodic motions of a free system are considered, provided that the pattern of motion is repeated over one or more intervals. If the periodicity extends over two intervals, then order of the system increases twofold. The system however remains linear and its solution can, basically, be obtained by analytic means.

Finally, simplest periodic motion of constrained systems can be investigated when

Finally, simplest periodic motion of constrained systems can be investigated when the spheres move in some (e. g. homogeneous) field of force with the result that conditions (2.1) no longer hold.

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ON THE MOTION EQUATIONS OF NONHOLONOMIC MECHANICAL SYSTEMS IN POINCARÉ-CHETAEV VARIABLES

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The Poincaré-Chetaev equations for holonomic mechanical systems have been written by Poincaré [1] and generalized by Chetaev to the dependent variables case [2]. The purpose of the present paper is to extend the mentioned method to the case of nonholonomic systems.

1. Formulation of the problem. Let us consider a nonholonomic mechanical system defined by the n Poincaré-Chetaev variables $x_1, \ldots, x_n$ [2], which are subject, in real displacements, to the following p holonomic and q nonholonomic constraints

$$a_{s1}x_1' + \ldots + a_{sn}x_n' + a_s = 0 \qquad (s = 1, \ldots, p) \tag{1.1}$$

$$\alpha_{\nu 1}x_1' + \ldots + \alpha_{\nu n}x_n' + \alpha_\nu = 0 \qquad (\nu = 1, \ldots, q) \tag{1.2}$$

and in possible displacements, to Eqs. [3]

$$a_{s1}\delta x_1 + \ldots + a_{sn}\delta x_n = 0 \qquad (s = 1, \ldots, p) \tag{1.3}$$

$$\alpha_{\nu 1}\delta x_1 + \ldots + \alpha_{\nu n}\delta x_n = 0 \qquad (\nu = 1, \ldots, q) \tag{1.4}$$

Here a_{si}, a_s, $\alpha_{\nu i}$, α_ν are functions of the time t and the variables $x_1, \ldots, x_n$; x_i' and δx_i are the derivatives and variations of the variables x_i . The constraints (1.1)